

# Stability of scaling regimes in $d \geq 2$ developed turbulence with weak anisotropy

M. Hnatic<sup>1</sup>, E. Jonyova<sup>1,2</sup>, M. Jurcisin<sup>1,2</sup>, M. Stehlik<sup>1</sup>

<sup>1</sup> Permanent address: *Institute of Experimental Physics, SAS, Košice, Slovakia* ,

<sup>2</sup> *Joint Institute for Nuclear Research, Dubna, Russia*

## Abstract

The fully developed turbulence with weak anisotropy is investigated by means of renormalization group approach (RG) and double expansion regularization for dimensions  $d \geq 2$ . Some modification of the standard minimal subtraction scheme has been used to analyze stability of the Kolmogorov scaling regime which is governed by the renormalization group fixed point. This fixed point is unstable at  $d = 2$ ; thus, the infinitesimally weak anisotropy destroys above scaling regime in two-dimensional space. The restoration of the stability of this fixed point, under transition from  $d = 2$  to  $d = 3$ , has been demonstrated at borderline dimension  $2 < d_c < 3$ . The results are in qualitative agreement with ones obtained recently in the framework of the usual analytical regularization scheme.

## 1 Introduction

A traditional approach to the description of fully developed turbulence is based on the stochastic Navier-Stokes equation [1]. The complexity of this equation leads to great difficulties which do not allow one to solve it even in the simplest case when one assumes the isotropy of the system under consideration. On the other hand, the isotropic turbulence is almost delusion and if exists is still rather rare. Therefore, if one wants to model more or less realistic developed turbulence, one is pushed to consider anisotropically forced turbulence rather than isotropic one. This, of course, rapidly increases complexity of the corresponding differential equation which itself has to involve the part responsible for description of the anisotropy. An exact solution of the stochastic Navier-Stokes equation does not exist and one is forced to find out some convenient methods to touch the problem at least step by step.

A suitable and also powerful tool in the theory of developed turbulence is the so-called renormalization group (RG) method<sup>1</sup>. Over the last two decades the RG

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<sup>1</sup> Here we consider the quantum-field renormalization group approach [2] rather than the Wilson renormalization group technique [3]

technique has widely been used in this field of science and gives answers to some principal questions (e.g. the fundamental description of the infra-red (IR) scale invariance) and is also useful for calculations of many universal parameters (e.g. critical dimensions of the fields and their gradients etc.). A detailed survey of this questions can be found in Refs. [4, 5] and Refs. therein.

In early papers, the RG approach has been applied only to the isotropic models of developed turbulence. However, the method can also be used (with some modifications) in the theory of anisotropic developed turbulence. A crucial question immediately arises here, whether the principal properties of the isotropic case and anisotropic one are the same at least at the qualitative level. If they are, then it is possible to consider the isotropic case as a first step in the investigation of real systems. On this way of transition from the isotropic developed turbulence into the anisotropic one we have to learn whether the scaling regime does remain stable under this transition. That means, whether the stable fixed points of the RG equations remain stable under the influence of anisotropy.

Over the last decade a few papers have appeared in which the above question has been considered. In some cases it has been found out that stability really takes place (see, e.g. [6, 7]). On the other hand, existence of systems without such a stability has been proved too. As has been shown in Ref. [8], in the anisotropic magnetohydrodynamic developed turbulence a stable regime generally does not exist. In [7, 9],  $d$ -dimensional models with  $d > 2$  were investigated for two cases: weak anisotropy [7] and strong one [9], and it has been shown that the stability of the isotropic fixed point is lost for dimensions  $d < d_c = 2.68$ . It has also been shown that stability of the fixed point even for dimension  $d = 3$  takes place only for sufficiently weak anisotropy. The only problem in these investigations is that it is impossible to use them in the case  $d = 2$  because new ultra-violet (UV) divergences appear in the Green functions when one considers  $d = 2$ , and they were not taken into account in [7, 9].

In [10], a correct treatment of the two-dimensional isotropic turbulence has been given. The correctness in the renormalization procedure has been reached by introduction into the model a new local term (with a new coupling constant) which allows one to remove additional UV-divergences. From this point of view, the results obtained earlier for anisotropical developed turbulence presented in [11] and based on the paper [12] (the results of the last paper are in conflict with [10]) cannot be considered as correct because they are inconsistent with the basic requirement of the UV-renormalization, namely with the requirement of the localness of the counterterms [13, 14].

The authors of the recent paper [15] have used the double-expansion procedure introduced in [10] (this procedure is a combination of the well-known Wilson dimensional regularization procedure and the analytical one) and the minimal subtraction (MS) scheme [16] for investigation of developed turbulence with weak anisotropy for  $d = 2$ . In such a perturbative approach the deviation of the spatial dimension from  $d = 2$ ,  $\delta = (d - 2)/2$ , and the deviation of the exponent of the powerlike correlation

function of random forcing from their critical values,  $\epsilon$ , play the role of the expansion parameters.

The main result of the paper was the conclusion that the two-dimensional fixed point is not stable under weak anisotropy. It means that 2d turbulence is very sensitive to the anisotropy and no stable scaling regimes exist in this case. In the case  $d = 3$ , for both the isotropic turbulence and anisotropic one, as it has been mentioned above, existence of the stable fixed point, which governs the Kolmogorov asymptotic regime, has been established by means of the RG approach by using the analytical regularization procedure [6, 7, 9]. One can make analytical continuation from  $d = 2$  to the three-dimensional turbulence (in the same sense as in the theory of critical phenomena) and verify whether the stability of the fixed point (or, equivalently, stability of the Kolmogorov scaling regime) is restored. From the analysis made in Ref. [15] it follows that it is impossible to restore the stable regime by transition from dimension  $d = 2$  to  $d = 3$ . We suppose that the main reason for the above described discrepancy is related to the straightforward application of the standard MS scheme. In the standard MS scheme one works with the purely divergent part of the Green functions only, and in concrete calculations its dependence on the space dimension  $d$  resulting from the tensor nature of these Green functions is neglected (see Sect.3). In the case of isotropic models, the stability of the fixed points is independent of dimension  $d$ . However, in anisotropic models the stability of fixed points depends on the dimension  $d$ , and consideration of the tensor structure of Feynman graphs in the analysis of their divergences becomes important.

In present paper we suggest applying modified MS scheme in which we keep the  $d$ -dependence of UV-divergences of graphs. We affirm that after such a modification the  $d$ -dependence is correctly taken into account and can be used in investigation of whether it is possible to restore the stability of the anisotropic developed turbulence for some dimension  $d_c$  when going from a two-dimensional system to a three-dimensional one. In the limit of infinitesimally weak anisotropy for the physically most reasonable value of  $\epsilon = 2$ , the value of the borderline dimension is  $d_c = 2.44$ . Below the borderline dimension, the stable regime of the fixed point of the isotropic developed turbulence is lost by influence of weak anisotropy.

It has to be mentioned that a similar idea of "geometric factor" was used in Ref. [17] in RG analysis of the Burgers-Kardar-Parisi-Zhang equation but the reason to keep the  $d$ -dependence of divergent parts of the graphs was to take correctly into account the finite part of one-loop Feynman diagrams in the two-loop approximation. In the present paper, we shall not discuss it in detail because the critical analysis of the results obtained in [17] was given in [18].

The paper is organized as follows. In Section 2 we give the quantum field functional formulation of the problem of the fully developed turbulence with weak anisotropy. The RG analysis is given in Section 3 when we discuss the stability of the fixed point obtained under weak anisotropy. In Section 4 we discuss our results. Appendix I contains expressions for the divergent parts of the important graphs. At the end, Appendix II contains analytical expressions for the fixed point and the

equation which describes its stability in the limit of the weak anisotropy.

## 2 Description of the Model. UV-divergencies

In this section we give the description of the model. As has already been discussed in the previous section, we work with fully developed turbulence and assume weak anisotropy of the system. It means that the parameters that describe deviations from the fully isotropic case are sufficiently small and allow one to forget about corrections of higher degrees (than linear) which are made by them.

In the statistical theory of anisotropically developed turbulence the turbulent flow can be described by a random velocity field  $\vec{v}(\vec{x}, t)$  and its evolution is given by the randomly forced Navier-Stokes equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} - \nu_0 \Delta \vec{v} - \vec{f}^A = \vec{f}, \quad (1)$$

where we assume incompressibility of the fluid, which is mathematically given by the well-known conditions  $\vec{\nabla} \cdot \vec{v} = 0$  and  $\vec{\nabla} \cdot \vec{f} = 0$ . In eq.(1) the parameter  $\nu_0$  is the kinematic viscosity (hereafter all parameters with subscript 0 denote bare parameters of unrenormalized theory, see below), the term  $\vec{f}^A$  is related to anisotropy and will be specified later. The large-scale random force per unit mass  $\vec{f}$  is assumed to have Gaussian statistics defined by the averages

$$\langle f_i \rangle = 0, \quad \langle f_i(\vec{x}_1, t) f_j(\vec{x}_2, t) \rangle = D_{ij}(\vec{x}_1 - \vec{x}_2, t_1 - t_2). \quad (2)$$

The two-point correlation matrix

$$D_{ij}(\vec{x}, t) = \delta(t) \int \frac{d^d \vec{k}}{(2\pi)^d} \tilde{D}_{ij}(\vec{k}) \exp(i\vec{k} \cdot \vec{x}) \quad (3)$$

is convenient to be parameterized in the following way [6, 8]:

$$\tilde{D}_{ij}(\vec{k}) = g_0 \nu_0^3 k^{4-d-2\epsilon} [(1 + \alpha_{10} \xi_k^2) P_{ij}(\vec{k}) + \alpha_{20} R_{ij}(\vec{k})], \quad (4)$$

where a vector  $\vec{k}$  is the wave vector,  $d$  is the dimension of the space (in our case:  $2 \leq d$ ),  $\epsilon \geq 0$  is a dimensionless parameter of the model. If the dimension of the system is taken  $d > 2$ , then the physical value of this parameter is  $\epsilon = 2$  (the so-called energy pumping regime). The situation is more complicated when  $d = 2$ . In this case the new integrals of motion arise, namely the enstrophy, and all its powers (for details see Ref.[19]) which leads to ambiguity in determination of the inertial range and this freedom is given in the RG method by the value of the parameter  $\epsilon$ . The value  $\epsilon = 3$  corresponds to the so-called enstrophy pumping regime. This problem of uncertainty cannot be solved in the framework of the RG technique. On the other hand, the value of  $\epsilon$  is not important for stability of the fixed point when

$d = 2$ . Thus, it is not important, from our point of view, what is the value of  $\epsilon$  in the case  $d = 2$ . Its value  $\epsilon = 0$  corresponds to a logarithmic perturbation theory for calculation of Green functions when  $g_0$ , which plays the role of a bare coupling constant of the model, becomes dimensionless. The problem of continuation from  $\epsilon = 0$  to physical values has been discussed in [20]. The  $(d \times d)$ -matrices  $P_{ij}$  and  $R_{ij}$  are the transverse projection operators and in the wave-number space are defined by the relations

$$P_{ij}(\vec{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}, \quad R_{ij}(\vec{k}) = \left( n_i - \xi_k \frac{k_i}{k} \right) \left( n_j - \xi_k \frac{k_j}{k} \right), \quad (5)$$

where  $\xi_k$  is given by the equation  $\xi_k = \vec{k} \cdot \vec{n}/k$ . In eq.(5) the unit vector  $\vec{n}$  specifies the direction of the anisotropy axis. The tensor  $\hat{D}_{ij}$  given by eq.(4) is the most general form with respect to the condition of incompressibility of the system under consideration and contains two dimensionless free parameters  $\alpha_{10}$  and  $\alpha_{20}$ . From the positiveness of the correlator tensor  $D_{ij}$  one immediately gets restrictions on the above parameters, namely  $\alpha_{10} \geq -1$  and  $\alpha_{20} \geq 0$ . In what follows we assume that these parameters are small enough and generate only small deviations from the isotropy case.

Using the well-known Martin-Siggia-Rose formalism of the stochastic quantization [21, 22, 23, 24] one can transform the stochastic problem (1) with the correlator (3) into the quantum field model of the fields  $\vec{v}$  and  $\vec{v}$ . Here  $\vec{v}$  is an independent of the  $\vec{v}$  auxiliary incompressible field which we have to introduce when transforming the stochastic problem into the functional form.

The action of the fields  $\vec{v}$  and  $\vec{v}$  is given in the form

$$S = \frac{1}{2} \int d^d \vec{x}_1 dt_1 d^d \vec{x}_2 dt_2 \left[ v_i(\vec{x}_1, t_1) D_{ij}(\vec{x}_1 - \vec{x}_2, t_1 - t_2) v_j(\vec{x}_2, t_2) \right] + \int d^d \vec{x} dt \left\{ \vec{v}(\vec{x}, t) \left[ -\partial_t \vec{v} - (\vec{v} \cdot \vec{\nabla}) \vec{v} + \nu_0 \vec{\nabla}^2 \vec{v} + \vec{f}^A \right] (\vec{x}, t) \right\}. \quad (6)$$

The functional formulation gives the possibility of using the quantum field theory methods including the RG technique to solve the problem. By means of the RG approach it is possible to extract large-scale asymptotic behaviour of the correlation functions after an appropriate renormalization procedure which is needed to remove UV-divergences.

Now we can return back to give an explicit form of the anisotropic dissipative term  $\vec{f}^A$ . When  $d > 2$  the UV-divergences are only present in the one-particle-irreducible Green function  $\langle \vec{v} \vec{v} \rangle$ . To remove them, one needs to introduce into the action in addition to the counterterm  $\vec{v} \cdot \vec{\nabla}^2 \vec{v}$  (the only counterterm needed in the isotropic model) the following ones  $\vec{v} \cdot (\vec{n} \cdot \nabla)^2 \vec{v}$ ,  $(\vec{n} \cdot \vec{v}) \vec{\nabla}^2 (\vec{n} \cdot \vec{v})$  and  $(\vec{n} \cdot \vec{v}) (\vec{n} \cdot \vec{\nabla})^2 (\vec{n} \cdot \vec{v})$ . These additional terms are needed to remove divergences related to anisotropic structures. In this case ( $d > 2$ ), one can use the above action (6) with (4) to solve the anisotropic turbulent problem. Therefore, in order to arrive at the multiplicative renormalizable

model, we have to take the term  $\vec{f}^A$  in the form

$$\vec{f}^A = \nu_0 \left[ \chi_{10}(\vec{n} \cdot \vec{\nabla})^2 \vec{v} + \chi_{20} \vec{n} \vec{\nabla}^2 (\vec{n} \cdot \vec{v}) + \chi_{30} \vec{n} (\vec{n} \cdot \vec{\nabla})^2 (\vec{n} \cdot \vec{v}) \right]. \quad (7)$$

Bare parameters  $\chi_{10}$ ,  $\chi_{20}$  and  $\chi_{30}$  characterize the weight of the individual structures in (7).

A more complicated situation arises in the specific case  $d = 2$  where new divergences appear. They are related to the 1-irreducible Green function  $\langle \vec{v} \vec{v} \rangle$  which is finite when  $d > 2$ . Here one comes to a problem how to remove these divergences because the term in our action, which contains a structure of this type is nonlocal, namely  $\vec{v} k^{4-d-2\epsilon} \vec{v}$ . The only correct way of solving the above problem is to introduce into the action a new local term of the form  $\vec{v} \vec{\nabla}^2 \vec{v}$  (isotropic case) [10]. In the anisotropic case, we have to introduce additional counterterms  $\vec{v} (\vec{n} \cdot \vec{\nabla})^2 \vec{v}$ ,  $(\vec{n} \cdot \vec{v}) \vec{\nabla}^2 (\vec{n} \cdot \vec{v})$  and  $(\vec{n} \cdot \vec{v}) (\vec{n} \cdot \vec{\nabla})^2 (\vec{n} \cdot \vec{v})$ . In [10, 12] a double-expansion method with a simultaneous deviation  $2\delta = d - 2$  from the spatial dimension  $d = 2$  and also a deviation  $\epsilon$  from the  $k^2$  form of the forcing pair correlation function proportional to  $k^{2-2\delta-2\epsilon}$  was proposed. We shall follow the formulation founded on the two-expansion parameters in the present paper.

In this case, the kernel (4) corresponding to the correlation matrix  $D_{ij}(\vec{x}_1 - \vec{x}_2, t_2 - t_1)$  in action (6) is replaced by the expression

$$\begin{aligned} \tilde{D}_{ij}(\vec{k}) &= g_{10} \nu_0^3 k^{2-2\delta-2\epsilon} [(1 + \alpha_{10} \xi_k^2) P_{ij}(\vec{k}) + \alpha_{20} R_{ij}(\vec{k})] \\ &+ g_{20} \nu_0^3 k^2 [(1 + \alpha_{30} \xi_k^2) P_{ij}(\vec{k}) + (\alpha_{40} + \alpha_{50} \xi_k^2) R_{ij}(\vec{k})]. \end{aligned} \quad (8)$$

Here  $P_{ij}$  and  $R_{ij}$  are given by relations (5),  $g_{20}$ ,  $\alpha_{30}$ ,  $\alpha_{40}$  and  $\alpha_{50}$  are new parameters of the model, and the parameter  $g_0$  in eq. (4) is now renamed as  $g_{10}$ . One can see that in such a formulation the counterterm  $\vec{v} \vec{\nabla}^2 \vec{v}$  and all anisotropic terms can be taken into account by renormalization of the coupling constant  $g_{20}$ , and the parameters  $\alpha_{30}$ ,  $\alpha_{40}$  and  $\alpha_{50}$ .

The action (6) with the kernel  $\tilde{D}_{ij}(\vec{k})$  (8) is given in the form convenient for realization of the quantum field perturbation analysis with the standard Feynman diagram technique. From the quadratic part of the action one obtains the matrix of bare propagators (in the wave-number - frequency representation)

$$\begin{aligned} \text{---} &= \langle v_i v_j \rangle_0 \equiv \Delta_{ij}^{vv}(\vec{k}, \omega_k), \\ \text{---} + &= \langle v_i v_j' \rangle_0 \equiv \Delta_{ij}^{vv'}(\vec{k}, \omega_k), \\ + \text{---} + &= \langle v_i' v_j' \rangle_0 \equiv \Delta_{ij}^{v'v'}(\vec{k}, \omega_k) = 0, \end{aligned}$$

where

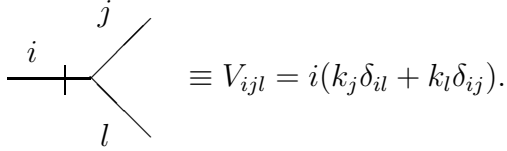
$$\Delta_{ij}^{vv}(\vec{k}, \omega_k) = -\frac{K_3}{K_1 K_2} P_{ij}$$

$$\begin{aligned}
& + \frac{1}{K_1(K_2 + \tilde{K}(1 - \xi_k^2))} \left[ \frac{\tilde{K}K_3}{K_2} + \frac{\tilde{K}(K_3 + K_4(1 - \xi_k^2))}{(K_1 + \tilde{K}(1 - \xi_k^2))} - K_4 \right] R_{ij} \\
\Delta_{ij}^{vv'}(\vec{k}, \omega_k) &= \frac{1}{K_2} P_{ij} - \frac{\tilde{K}}{K_2(K_2 + \tilde{K}(1 - \xi_k^2))} R_{ij},
\end{aligned} \tag{9}$$

with

$$\begin{aligned}
K_1 &= i\omega_k + \nu_0 k^2 + \nu_0 \chi_{10} (\vec{n} \cdot \vec{k})^2, \\
K_2 &= -i\omega_k + \nu_0 k^2 + \nu_0 \chi_{10} (\vec{n} \cdot \vec{k})^2, \\
K_3 &= -g_{10} \nu_0^3 k^{2-2\delta-2\epsilon} (1 + \alpha_{10} \xi_k^2) - g_{20} \nu_0^3 k^2 (1 + \alpha_{30} \xi_k^2), \\
K_4 &= -g_{10} \nu_0^3 k^{2-2\delta-2\epsilon} \alpha_{20} - g_{20} \nu_0^3 k^2 (\alpha_{40} + \alpha_{50} \xi_k^2), \\
\tilde{K} &= \nu_0 \chi_{20} k^2 + \nu_0 \chi_{30} (\vec{n} \cdot \vec{k})^2.
\end{aligned} \tag{10}$$

The propagators are written in the form suitable also for strong anisotropy when the parameters  $\alpha_{i0}$  are not small. In the case of weak anisotropy, it is possible to make the expansion and work only with linear terms with respect to all parameters which characterize anisotropy. The interaction vertex in our model is given by the expression



$$\equiv V_{ijl} = i(k_j \delta_{il} + k_l \delta_{ij}).$$

Here, the wave vector  $\vec{k}$  corresponds to the field  $\vec{v}$ . Now one can use the above introduced Feynman rules for computation of all needed graphs.

### 3 RG-analysis and Stability of the Fixed Point

Using the standard analysis of quantum field theory (see e.g. [4, 5, 13, 14]), one can find out that the UV divergences of one-particle-irreducible Green functions  $\langle vv \rangle_{IR}$  and  $\langle vv \rangle_{IR}$  are quadratic in the wave vector. The last one takes place only in the case when dimension of the space is two. All terms needed for removing the divergences are included in the action (6) with (7) and kernel (8). This leads to the fact that our model is multiplicatively renormalizable. Thus, one can immediately write down the renormalized action in wave-number - frequency representation with  $\vec{\nabla} \rightarrow i\vec{k}$ ,  $\partial_t \rightarrow -i\omega_k$  (all needed integrations and summations are assumed)

$$S^R(v, v) = \frac{1}{2} v_i \left[ g_1 \nu^3 \mu^{2\epsilon} k^{2-2\delta-2\epsilon} \left( (1 + \alpha_1 \xi_k^2) P_{ij} + \alpha_2 R_{ij} \right) \right]$$

$$\begin{aligned}
& + g_2 \nu^3 \mu^{-2\delta} k^2 \left( (Z_5 + Z_6 \alpha_3 \xi_k^2) P_{ij} + (Z_7 \alpha_4 + Z_8 \alpha_5 \xi_k^2) R_{ij} \right) \Big] v_j \\
& + v_i \left[ (i\omega_k - Z_1 \nu k^2) P_{ij} - \nu k^2 \left( Z_2 \chi_1 \xi_k^2 P_{ij} + (Z_3 \chi_2 + Z_4 \chi_3 \xi_k^2) R_{ij} \right) \right] v_j \\
& + \frac{1}{2} v_i v_j v_l V_{ijl}, \tag{11}
\end{aligned}$$

where  $\mu$  is a scale setting parameter with the same canonical dimension as the wave number. Quantities  $g_i$ ,  $\chi_i$ ,  $\alpha_3$ ,  $\alpha_4$ ,  $\alpha_5$  and  $\nu$  are the renormalized counterparts of bare ones and  $Z_i$  are renormalization constants which are expressed via the UV divergent parts of the functions  $\langle vv \rangle_{IR}$  and  $\langle vv \rangle_{IR}$ . Their general form in one loop approximation is

$$Z_i = 1 - F_i \text{Poles}_i^{\delta, \epsilon}. \tag{12}$$

In the standard MS scheme the amplitudes  $F_i$  are only some functions of  $g_i$ ,  $\chi_i$ ,  $\alpha_3$ ,  $\alpha_4$ ,  $\alpha_5$  and are independent of  $d$  and  $\epsilon$ . The terms  $\text{Poles}_i^{\delta, \epsilon}$  are given by linear combinations of the poles  $\frac{1}{\epsilon}$ ,  $\frac{1}{\delta}$  and  $\frac{1}{2\epsilon+\delta}$  (for  $\delta \rightarrow 0, \epsilon \rightarrow 0$ ). The amplitudes  $F_i = F_i^1 F_i^2$  are a product of two multipliers  $F_i^1, F_i^2$ . One of them, say,  $F_i^1$  is a multiplier originating from the divergent part of the Feynman diagrams, and the second one  $F_i^2$  is connected only with the tensor nature of the diagrams. We explain that using the following simple example. Consider an UV-divergent integral

$$I(\mathbf{k}, \mathbf{n}) \equiv n_i n_j k_l k_m \int d^d \mathbf{q} \frac{1}{(q^2 + m^2)^{1+2\delta}} \left( \frac{q_i q_j q_l q_m}{q^4} - \frac{\delta_{ij} q_l q_m + \delta_{il} q_j q_m + \delta_{jl} q_i q_m}{3q^2} \right)$$

(summations over repeated indices are implied) where  $m$  is an infrared mass. It can be simplified in the following way:

$$I(\mathbf{k}, \mathbf{n}) \equiv n_i n_j k_l k_m S_{ijlm} \int_0^\infty dq^2 \frac{q^{2\delta}}{2(q^2 + m^2)^{1+2\delta}},$$

where

$$S_{ijlm} = \frac{S_d}{d(d+2)} (\delta_{ij} \delta_{lm} + \delta_{il} \delta_{jm} + \delta_{im} \delta_{jl} - \frac{(d+2)}{3} (\delta_{ij} \delta_{lm} + \delta_{il} \delta_{jm} + \delta_{im} \delta_{jl})),$$

$$\int_0^\infty dq^2 \frac{q^{2\delta}}{2(q^2 + m^2)^{1+2\delta}} = \frac{\Gamma(\delta+1)\Gamma(\delta)}{2m^{2\delta}\Gamma(2\delta+1)},$$

and  $S_d = 2\pi^{d/2}/\Gamma(d/2)$  is the surface of unit the d-dimensional sphere. The purely UV divergent part manifests itself as the pole in  $2\delta = d - 2$ ; therefore, we find

$$\text{UV div. part of } I = \frac{1}{2\delta} (F_1^2 k^2 + F_2^2 (\mathbf{n}\mathbf{k})^2),$$

where  $F_1^2 = F_2^2/2 = (1-d)S_d/3d(d+2)$ .



The transition from the action (6) to the renormalized one (11) is given by the introduction of the following renormalization constants  $Z$ :

where  $i = 1, 2, 3$ . By comparison of the corresponding terms in the action (11) with definitions of the renormalization constants  $Z$  for the parameters (13), one can immediately write down relations between them. Namely, we have

where again  $i = 1, 2, 3$ .

$$\langle v'v \rangle_{IR} = \text{diagram with a bubble and two thick external lines}, \quad \langle v'v \rangle_{IR} = \text{diagram with a bubble and one thick external line and one thin external line}. \quad (15)$$
$$\begin{aligned} \Gamma^{v'v'} &= \frac{1}{2} \nu^3 A \\ &\times \left[ \frac{g_1^2}{4\epsilon + 2\delta} \left( a_1 \delta_{ij} k^2 + a_2 \delta_{ij} (\vec{n} \cdot \vec{k})^2 + a_3 n_i n_j k^2 + a_4 n_i n_j (\vec{n} \cdot \vec{k})^2 \right) \right. \\ &+ \frac{g_1 g_2}{2\epsilon} \left( b_1 \delta_{ij} k^2 + b_2 \delta_{ij} (\vec{n} \cdot \vec{k})^2 + b_3 n_i n_j k^2 + b_4 n_i n_j (\vec{n} \cdot \vec{k})^2 \right) \\ &\left. + \frac{g_2^2}{-2\delta} \left( c_1 \delta_{ij} k^2 + c_2 \delta_{ij} (\vec{n} \cdot \vec{k})^2 + c_3 n_i n_j k^2 + c_4 n_i n_j (\vec{n} \cdot \vec{k})^2 \right) \right], \end{aligned}$$

$$\begin{aligned}
\Gamma^{v,v} &= -\nu A \\
&\times \left[ \frac{g_1}{2\epsilon} (d_1 \delta_{ij} k^2 + d_2 \delta_{ij} (\vec{n} \cdot \vec{k})^2 + d_3 n_i n_j k^2 + d_4 n_i n_j (\vec{n} \cdot \vec{k})^2) \right. \\
&\left. + \frac{g_2}{-2\delta} (e_1 \delta_{ij} k^2 + e_2 \delta_{ij} (\vec{n} \cdot \vec{k})^2 + e_3 n_i n_j k^2 + e_4 n_i n_j (\vec{n} \cdot \vec{k})^2) \right], \quad (16)
\end{aligned}$$

where parameter  $A$  and functions  $a_i$ ,  $b_i$ ,  $c_i$ ,  $d_i$  and  $e_i$  are given in the Appendix I ( $i = 1 \dots 4$ ). The counterterms are built up from these divergent parts which lead to the following equations for renormalization constants:

$$\begin{aligned}
Z_1 &= 1 - A \left( \frac{g_1}{2\epsilon} d_1 - \frac{g_2}{2\delta} e_1 \right), \\
Z_{1+i} &= 1 - \frac{A}{\chi_i} \left( \frac{g_1}{2\epsilon} d_{1+i} - \frac{g_2}{2\delta} e_{1+i} \right), \\
Z_5 &= 1 + \frac{A}{2} \left( \frac{g_1^2}{g_2} \frac{a_1}{4\epsilon + 2\delta} + \frac{g_1}{2\epsilon} b_1 - \frac{g_2}{2\delta} c_1 \right), \\
Z_{5+i} &= 1 + \frac{A}{2\alpha_{i+2}} \left( \frac{g_1^2}{g_2} \frac{a_{i+1}}{4\epsilon + 2\delta} + \frac{g_1}{2\epsilon} b_{i+1} - \frac{g_2}{2\delta} c_{i+1} \right), \\
i &= 1, 2, 3. \quad (17)
\end{aligned}$$

From these expressions one can define the corresponding anomalous dimensions  $\gamma_i = \mu \partial_\mu \ln Z_i$  for all renormalization constants  $Z_i$  (logarithmic derivative  $\mu \partial_\mu$  is taken at fixed values of all bare parameters). The  $\beta$ -functions for all invariant charges (running coupling constants  $g_1$ ,  $g_2$ , and parameters  $\chi_i$ ,  $\alpha_{i+2}$ ) where  $i = 1, 2, 3$  are given by the following relations:  $\beta_{g_i} = \mu \partial_\mu g_i$  ( $i = 1, 2$ ),  $\beta_{\chi_i} = \mu \partial_\mu \chi_i$  and  $\beta_{\alpha_{i+2}} = \mu \partial_\mu \alpha_{i+2}$  ( $i = 1, 2, 3$ ). Now using equations (14) and definitions given above one can immediately write the  $\beta$ -functions in the form

$$\begin{aligned}
\beta_{g_1} &= -g_1(2\epsilon + \gamma_{g_1}) = g_1(-2\epsilon + 3\gamma_1), \\
\beta_{g_2} &= g_2(2\delta - \gamma_{g_2}) = g_2(2\delta + 3\gamma_1 - \gamma_5), \\
\beta_{\chi_i} &= -\chi_i \gamma_{\chi_i} = -\chi_i(\gamma_{i+1} - \gamma_1), \\
\beta_{\alpha_{i+2}} &= -\alpha_{i+2} \gamma_{\alpha_{i+2}} = -\alpha_{i+2}(\gamma_{i+5} - \gamma_5), \quad i = 1, 2, 3, \quad (18)
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1 &= A(g_1 d_1 + g_2 e_1), \\
\gamma_{i+1} &= \frac{A}{\chi_i} (g_1 d_{i+1} + g_2 e_{i+1}), \\
\gamma_5 &= -\frac{A}{2} \left( \frac{g_1^2}{g_2} a_1 + g_1 b_1 + g_2 c_1 \right), \\
\gamma_{i+5} &= -\frac{A}{2\alpha_{i+2}} \left( \frac{g_1^2}{g_2} a_{i+1} + g_1 b_{i+1} + g_2 c_{i+1} \right), \quad i = 1, 2, 3. \quad (19)
\end{aligned}$$

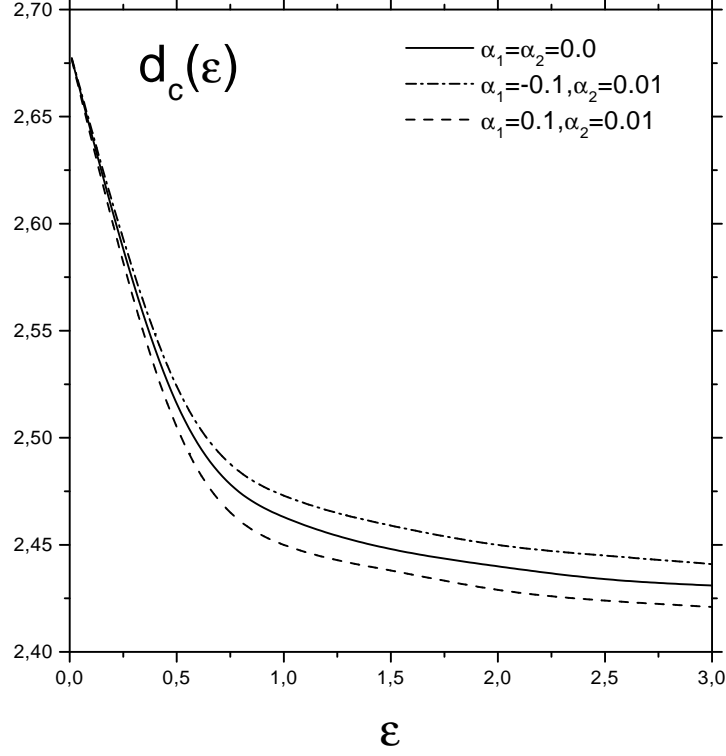


Figure 1: Dependence of the borderline dimension  $d_c$  on the parameter  $\epsilon$  and for concrete values of  $\alpha_1$  and  $\alpha_2$ .

By substitution of the anomalous dimensions  $\gamma_i$  (19) into the expressions for the  $\beta$ -functions one obtains

$$\begin{aligned}
\beta_{g_1} &= g_1(-2\epsilon + 3A(g_1d_1 + g_2e_1)), \\
\beta_{g_2} &= g_2 \left[ 2\delta + 3A(g_1d_1 + g_2e_1) + \frac{A}{2} \left( \frac{g_1^2}{g_2}a_1 + g_1b_1 + g_2c_1 \right) \right], \\
\beta_{\chi_i} &= -A[(g_1d_{i+1} + g_2e_{i+1}) - \chi_i(g_1d_1 + g_2e_1)], \\
\beta_{\alpha_{i+2}} &= -\frac{A}{2} \left[ - \left( \frac{g_1^2}{g_2}a_{i+1} + g_1b_{i+1} + g_2c_{i+1} \right) + \alpha_{i+2} \left( \frac{g_1^2}{g_2}a_1 + g_1b_1 + g_2c_1 \right) \right], \\
i &= 1, 2, 3.
\end{aligned} \tag{20}$$

The fixed point of the RG-equations is defined by the system of eight equations

$$\beta_C(C_*) = 0, \tag{21}$$

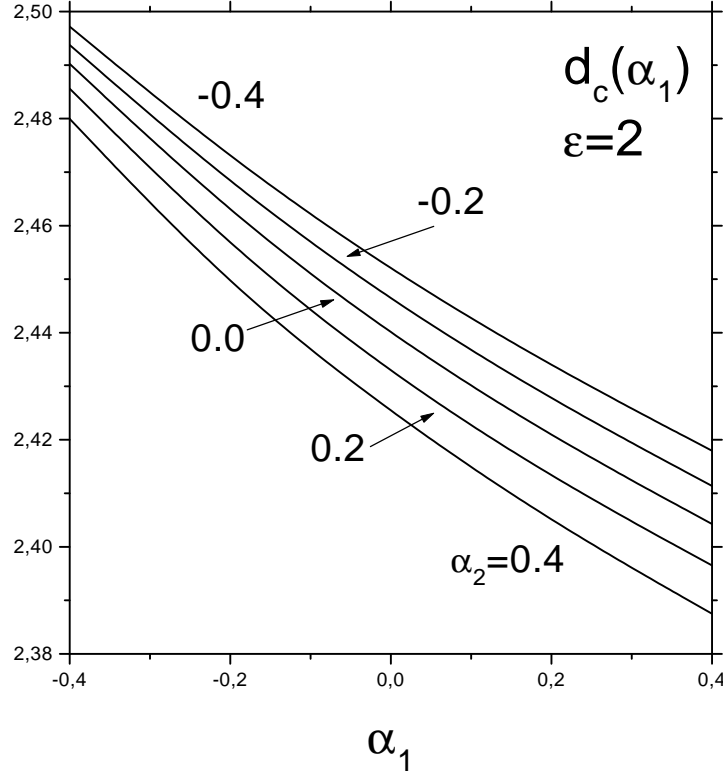


Figure 2: Dependence of the borderline dimension  $d_c$  on the parameters  $\alpha_1$  and  $\alpha_2$  for physical value  $\epsilon = 2$ .

where we denote  $C = \{g_1, g_2, \chi_i, \alpha_{i+2}\}$ ,  $i = 1, 2, 3$  and  $C_*$  is the corresponding value for the fixed point. The IR stability of the fixed point is determined by the positive real parts of the eigenvalues of the matrix

$$\omega_{lm} = \left( \frac{\partial \beta_{C_l}}{\partial C_m} \right)_{C=C_*} \quad l, m = 1, \dots, 8. \quad (22)$$

Now we have all necessary tools at hand to investigate the fixed points and their stability. In *the isotropic case* all parameters which are connected with the anisotropy are equal to zero, and one can immediately find the Kolmogorov fixed point, namely:

$$\begin{aligned} g_{1*} &= \frac{1}{A} \frac{8(2+d)\epsilon(2\epsilon - 3d(\delta + \epsilon) + d^2(3\delta + 2\epsilon))}{9(-1+d)^3d(1+d)(\delta + \epsilon)}, \\ g_{2*} &= \frac{1}{A} \frac{8(-4 - 2d + 2d^2 + d^3)\epsilon^2}{9(-1+d)^3d(1+d)(\delta + \epsilon)}, \end{aligned} \quad (23)$$

where  $\delta = (d - 2)/2$  and the corresponding  $\omega_{ij}$  matrix has the following eigenvalues:

$$\begin{aligned} \lambda_{1,2} = & \frac{1}{6d(d-1)} \left\{ 6d\delta(d-1) + 4\epsilon(2-3d+2d^2) \right. \\ & \pm \left[ (6d\delta(1-d) - 4\epsilon(2-3d+2d^2))^2 \right. \\ & \left. \left. - 12d(d-1)\epsilon(12d\delta(d-1) + 4\epsilon(2-3d+2d^2)) \right]^{\frac{1}{2}} \right\}. \end{aligned} \quad (24)$$

By a detail analysis of these eigenvalues we know that in the interesting region of parameters, namely  $\epsilon > 0$  and  $\delta \geq 0$  (it corresponds to  $d \geq 2$ ) the above computed fixed point is stable. In the limit  $d = 2$ , this fixed point is in agreement with that given in [10, 15].

When one considers *the weak anisotropy case* the situation becomes more complicated because of necessity to use all system of  $\beta$ -functions if one wants to analyze the stability of the fixed point. It is also possible to find analytical expressions for the fixed point in this more complicated case because in the weak anisotropy limit it is enough to calculate linear corrections of  $\alpha_1$  and  $\alpha_2$  to all the quantities (see Appendix II).

To investigate the stability of the fixed point it is necessary to apply it in the matrix (22). Analysis of this matrix shows us that it can be written in the block-diagonal form:  $(6 \times 6)(2 \times 2)$ . The  $(2 \times 2)$  part is given by the  $\beta$ -functions of the parameters  $\alpha_5$  and  $\chi_3$  and, namely, this block is responsible for the existence of the borderline dimension  $d_c$  because one of its eigenvalues, say  $\lambda_1(\epsilon, d, \alpha_1, \alpha_2)$ , has a solution  $d_c \in \langle 2, 3 \rangle$  of the equation  $\lambda_1(\epsilon, d_c, \alpha_1, \alpha_2) = 0$  for the defined values of  $\epsilon, \alpha_1, \alpha_2$ . The following procedure has been used to find the fixed point: First we have used the isotropic solution to  $g_1$  and  $g_2$  to compute the expressions for  $\alpha_{i+2}$  and  $\chi_i$ ,  $i = 1, 2, 3$ . From equations  $\beta_{\alpha_5} = 0$  and  $\beta_{\chi_3} = 0$  one can immediately find that  $\alpha_{5*} = 0$  and  $\chi_{3*} = 0$ . After this we can calculate expressions for the fixed point of the parameters  $\alpha_{i+2}$  and  $\chi_i$ ,  $i = 1, 2$ . At the end, we come back to the equations for  $g_1$  and  $g_2$ , namely  $\beta_{g_1} = 0$  and  $\beta_{g_2} = 0$ , and find linear corrections of  $\alpha_1$  and  $\alpha_2$  to the fixed point. The corresponding expressions for the fixed point and the corresponding eigenvalue of the stability matrix responsible for instability are given in Appendix II.

From a numerical analysis of the stability matrix one can find that in some region of space dimensions  $d$  the stability is lost by the influence of the weak anisotropy. On the other hand, the borderline dimension  $d_c$  arises when going from dimension  $d = 2$  to  $d = 3$ . For the energy pumping regime ( $\epsilon = 2$ ) we found the critical dimension  $d_c = 2.44$ . This value corresponds to  $\alpha_1 = \alpha_2 = 0$ . This is the case when one supposes only the fact of anisotropy. Using nonzero values of  $\alpha_1$  and  $\alpha_2$  one can also estimate the influence of these parameters on the borderline dimension  $d_c$ . It is interesting to calculate the dependence of  $d_c$  on the parameter  $\epsilon$  too. In Fig. 1, this dependence and the dependence on small values of  $\alpha_1$  and  $\alpha_2$  are presented. As one

can see from this figure  $d_c$  increases when  $\epsilon \rightarrow 0$  and also the parameters  $\alpha_1$  and  $\alpha_2$  give small corrections to  $d_c$ . In Fig. 2, the dependence of  $d_c$  on  $\alpha_1$  and  $\alpha_2$  for  $\epsilon = 2$  is presented.

## 4 Conclusion

We have investigated the influence of the weak anisotropy on the fully developed turbulence using the quantum field RG double expansion method and introduced the modified minimal subtraction scheme in which the space dimension dependence of the divergent parts of the Feynman diagrams is kept. We affirm that such a modified approach is correct when one needs to compute the  $d$ -dependence of the important quantities and is necessary for restoration of the stability of scaling regimes when one makes transition from dimension  $d = 2$  to  $d = 3$ . We have derived analytical expressions for the fixed point in the limit of the weak anisotropy and found the equation which manages the stability of this point as a function of the parameters  $\epsilon, \alpha_1$  and  $\alpha_2$ , and allows one to calculate the borderline dimension  $d_c$ . Below this dimension the fixed point is unstable. In the limit case of infinitesimally small anisotropy ( $\alpha_1 \rightarrow 0$  and  $\alpha_2 \rightarrow 0$ ) and in the energy pumping regime ( $\epsilon = 2$ ) we have found the borderline dimension  $d_c = 2.44$ . We have also investigated the  $\epsilon$ -dependence of  $d_c$  for different values of the anisotropy parameters  $\alpha_1, \alpha_2$  and also the dependence of  $d_c$  on the relatively small values of  $\alpha_1$  and  $\alpha_2$  for the physical value  $\epsilon = 2$ .

## Acknowledgements

This work was partially supported by Slovak Academy of Sciences within the project 7232.

## Appendix I

The explicit form of the parameter  $A$  and functions  $a_i, b_i, c_i, d_i$  and  $e_i$  ( $i = 1...4$ ) for the divergent parts of diagrams (15)

$$\begin{aligned}
a_1 &= \frac{1}{2d(2+d)(4+d)(6+d)} \\
&\times [-48 - 20d + 70d^2 + 30d^3 - 21d^4 - 10d^5 - d^6 \\
&+ \alpha_2(24 + 16d - 22d^2 - 16d^3 - 2d^4) + \alpha_1(24 + 52d - 4d^2 - 50d^3 - 20d^4 - 2d^5) \\
&+ \chi_1(-36 - 78d + 6d^2 + 75d^3 + 30d^4 + 3d^5) \\
&+ \chi_2(-36 - 24d + 33d^2 + 24d^3 + 3d^4) + \chi_3(-36 - 9d + 36d^2 + 9d^3)], \\
a_2 &= \frac{1}{4d(2+d)(4+d)(6+d)}
\end{aligned}$$

$$\begin{aligned}
& \times [\alpha_1(-96 - 64d + 88d^2 + 64d^3 + 8d^4) \\
& + \alpha_2(-96 - 64d + 124d^2 + 82d^3 - 26d^4 - 18d^5 - 2d^6) \\
& + \chi_1(144 + 96d - 132d^2 - 96d^3 - 12d^4) \\
& + \chi_2(144 + 96d - 186d^2 - 123d^3 + 39d^4 + 27d^5 + 3d^6) \\
& + \chi_3(72 + 6d - 87d^2 - 9d^3 + 15d^4 + 3d^5)], \\
a_3 &= a_2, \\
a_4 &= \frac{6\chi_3(1 - d^2)}{(2 + d)(6 + d)}, \\
b_1 &= \frac{1}{d(2 + d)(4 + d)(6 + d)} \\
& \times [-48 - 20d + 70d^2 + 30d^3 - 21d^4 - 10d^5 - d^6 + \alpha_5(12 + 3d - 12d^2 - 3d^3) \\
& + (\alpha_2 + \alpha_4)(12 + 8d - 11d^2 - 8d^3 - d^4) \\
& + (\alpha_1 + \alpha_3)(12 + 26d - 2d^2 - 25d^3 - 10d^4 - d^5) \\
& + \chi_1(-36 - 78d + 6d^2 + 75d^3 + 30d^4 + 3d^5) \\
& + \chi_2(-36 - 24d + 33d^2 + 24d^3 + 3d^4) + \chi_3(-36 - 9d + 36d^2 + 9d^3)], \\
b_2 &= \frac{1}{2d(2 + d)(4 + d)(6 + d)} \\
& \times [(\alpha_1 + \alpha_3)(-48 - 32d + 44d^2 + 32d^3 + 4d^4) \\
& + \alpha_5(-24 - 2d + 29d^2 + 3d^3 - 5d^4 - d^5) \\
& + (\alpha_2 + \alpha_4)(-48 - 32d + 62d^2 + 41d^3 - 13d^4 - 9d^5 - d^6) \\
& + \chi_1(144 + 96d - 132d^2 - 96d^3 - 12d^4) \\
& + \chi_2(144 + 96d - 186d^2 - 123d^3 + 39d^4 + 27d^5 + 3d^6) \\
& + \chi_3(72 + 6d - 87d^2 - 9d^3 + 15d^4 + 3d^5)], \\
b_3 &= b_2, \\
b_4 &= \frac{4(d^2 - 1)(\alpha_5 - 3\chi_3)}{(2 + d)(6 + d)}, \\
c_1 &= \frac{1}{2d(2 + d)(4 + d)(6 + d)} \\
& \times [-48 - 20d + 70d^2 + 30d^3 - 21d^4 - 10d^5 - d^6 \\
& + \alpha_5(24 + 6d - 24d^2 - 6d^3) + \alpha_4(24 + 16d - 22d^2 - 16d^3 - 2d^4) \\
& + \alpha_3(24 + 52d - 4d^2 - 50d^3 - 20d^4 - 2d^5) \\
& + \chi_1(-36 - 78d + 6d^2 + 75d^3 + 30d^4 + 3d^5) \\
& + \chi_2(-36 - 24d + 33d^2 + 24d^3 + 3d^4) + \chi_3(-36 - 9d + 36d^2 + 9d^3)], \\
c_2 &= \frac{1}{4d(2 + d)(4 + d)(6 + d)} \\
& \times [\alpha_3(-96 - 64d + 88d^2 + 64d^3 + 8d^4) \\
& + \alpha_5(-48 - 4d + 58d^2 + 6d^3 - 10d^4 - 2d^5)
\end{aligned}$$

$$\begin{aligned}
& + \alpha_4(-96 - 64d + 124d^2 + 82d^3 - 26d^4 - 18d^5 - 2d^6) \\
& + \chi_1(144 + 96d - 132d^2 - 96d^3 - 12d^4) \\
& + \chi_2(144 + 96d - 186d^2 - 123d^3 + 39d^4 + 27d^5 + 3d^6) \\
& + \chi_3(72 + 6d - 87d^2 - 9d^3 + 15d^4 + 3d^5)], \\
c_3 &= c_2, \\
c_4 &= \frac{(d^2 - 1)(4\alpha_5 - 6\chi_3)}{(2 + d)(6 + d)}, \\
d_1 &= \frac{1}{4d(2 + d)(4 + d)(6 + d)} \\
& \times [24d - 14d^2 - 33d^3 + 13d^4 + 9d^5 + d^6 + \alpha_2(12 - 4d - 13d^2 + 4d^3 + d^4) \\
& + \alpha_1(-12 - 20d + 3d^2 + 19d^3 + 9d^4 + d^5) \\
& + \chi_1(36 + 42d - 18d^2 - 40d^3 - 18d^4 - 2d^5) \\
& + \chi_2(-12 + 16d + 15d^2 - 16d^3 - 3d^4) + \chi_3(6 + 9d - 6d^2 - 9d^3)], \\
d_2 &= \frac{1}{8d(2 + d)(4 + d)(6 + d)} \\
& \times [\alpha_1(-48 + 16d + 52d^2 - 16d^3 - 4d^4) \\
& + \alpha_2(48 + 80d - 60d^2 - 96d^3 + 10d^4 + 16d^5 + 2d^6) \\
& + \chi_1(48 - 64d - 60d^2 + 64d^3 + 12d^4) \\
& + \chi_2(-48 - 104d + 62d^2 + 127d^3 - 11d^4 - 23d^5 - 3d^6) \\
& + \chi_3(-2d + 7d^2 + 5d^3 - 7d^4 - 3d^5)], \\
d_3 &= \frac{1}{8d(2 + d)(4 + d)(6 + d)} \\
& \times [\alpha_1(48 + 56d - 40d^2 - 56d^3 - 8d^4) + \alpha_2(-48 - 56d + 40d^2 + 56d^3 + 8d^4) \\
& + \chi_1(-48 - 104d + 32d^2 + 104d^3 + 16d^4) \\
& + \chi_2(48 + 32d - 38d^2 - 25d^3 - 9d^4 - 7d^5 - d^6) \\
& + \chi_3(22d - d^2 - 21d^3 + d^4 - d^5)], \\
d_4 &= \frac{\chi_3(-10 + d + 10d^2 - d^3)}{2(2 + d)(6 + d)}, \\
e_1 &= \frac{1}{4d(2 + d)(4 + d)(6 + d)} \\
& \times [24d - 14d^2 - 33d^3 + 13d^4 + 9d^5 + d^6 + 3d\alpha_5(-1 + d^2) \\
& + \alpha_4(12 - 4d - 13d^2 + 4d^3 + d^4) \\
& + \alpha_3(-12 - 20d + 3d^2 + 19d^3 + 9d^4 + d^5) \\
& + \chi_1(36 + 42d - 18d^2 - 40d^3 - 18d^4 - 2d^5) \\
& + \chi_2(-12 + 16d + 15d^2 - 16d^3 - 3d^4) + \chi_3(6 + 9d - 6d^2 - 9d^3)], \\
e_2 &= \frac{1}{8d(2 + d)(4 + d)(6 + d)}
\end{aligned}$$



$$\begin{aligned}
& \times [\alpha_3(-48 + 16d + 52d^2 - 16d^3 - 4d^4) + \alpha_5(-8d^2 - 2d^3 + 8d^4 + 2d^5) \\
& + \alpha_4(48 + 80d - 60d^2 - 96d^3 + 10d^4 + 16d^5 + 2d^6) \\
& + \chi_1(48 - 64d - 60d^2 + 64d^3 + 12d^4) \\
& + \chi_2(-48 - 104d + 62d^2 + 127d^3 - 11d^4 - 23d^5 - 3d^6) \\
& + \chi_3(-2d + 7d^2 + 5d^3 - 7d^4 - 3d^5)], \\
e_3 &= \frac{1}{8d(2+d)(4+d)(6+d)} \\
& \times [24d\alpha_5(-1 + d^2) + \alpha_3(48 + 56d - 40d^2 - 56d^3 - 8d^4) \\
& + \alpha_4(-48 - 56d + 40d^2 + 56d^3 + 8d^4) \\
& + \chi_1(-48 - 104d + 32d^2 + 104d^3 + 16d^4) \\
& + \chi_2(48 + 32d - 38d^2 - 25d^3 - 9d^4 - 7d^5 - d^6) \\
& + \chi_3(22d - d^2 - 21d^3 + d^4 - d^5)], \\
e_4 &= \frac{6\alpha_5(1 - d^2) + \chi_3(-10 + d + 10d^2 - d^3)}{2(2+d)(6+d)}, \\
A &= \frac{S_d}{(2\pi)^d(d^2 - 1)},
\end{aligned}$$

where  $S_d$  is  $d$ -dimensional sphere given by the following relation:

$$S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}.$$

## Appendix II

We present here the explicit analytical expressions for the fixed point in the weak anisotropy limit and also the equation which governs its stability.

The basic form of the fixed point is

$$\begin{aligned}
g_{1*} &= g_{10*} + g_{11*}\alpha_1 + g_{12*}\alpha_2, \\
g_{2*} &= g_{20*} + g_{21*}\alpha_1 + g_{22*}\alpha_2, \\
\alpha_{3*} &= e_{11}\alpha_1 + e_{12}\alpha_2, \\
\alpha_{4*} &= e_{21}\alpha_1 + e_{22}\alpha_2, \\
\chi_{1*} &= e_{31}\alpha_1 + e_{32}\alpha_2, \\
\chi_{2*} &= e_{41}\alpha_1 + e_{42}\alpha_2, \\
\alpha_{5*} &= 0, \\
\chi_{3*} &= 0,
\end{aligned}$$

where  $g_{10*}$  and  $g_{20*}$  are defined in eq.(23), and  $g_{11*}, g_{12*}, g_{21*}, g_{22*}$  and  $e_{ij}, i = 1, 2, 3, 4, j = 1, 2$  are functions only of the dimension  $d$  and parameters  $\epsilon$  and  $\delta = (d - 2)/2$ .

They have the following form:

$$\begin{aligned}
g_{11*} &= \frac{g_{11n}}{g_{11d}}, g_{12*} = \frac{g_{12n}}{g_{12d}}, g_{21*} = \frac{g_{21n}}{g_{22d}}, g_{22*} = \frac{g_{22n}}{g_{22d}}, \\
e_{11} &= \frac{e_{11n}}{e_d}, e_{12} = \frac{e_{12n}}{e_d}, e_{21} = \frac{e_{21n}}{e_d}, e_{22} = \frac{e_{22n}}{e_d}, \\
e_{31} &= \frac{e_{31n}}{g_s e_d}, e_{32} = \frac{e_{32n}}{g_s e_d}, e_{41} = \frac{e_{41n}}{g_s e_d}, e_{42} = \frac{e_{42n}}{g_s e_d},
\end{aligned}$$

where

$$\begin{aligned}
g_{11n} &= 3(d^2 - 1)g_{10*}(d^6(g_{10*} + g_{20*})((5e_{31} - 3)g_{10*} - 3e_{11}g_{20*} + 5e_{31}g_{20*}) \\
&+ 3d^5(g_{10*} + g_{20*})((-2 + 3e_{31} + 2e_{41})g_{10*} - (2e_{11} + e_{21} - 3e_{31} - 2e_{41})g_{20*}) \\
&- 8(g_{10*} + g_{20*})((-1 + 3e_{31} - e_{41})g_{10*} - (e_{11} - e_{21} - 3e_{31} + e_{41})g_{20*}) \\
&+ d^3(-((g_{10*} + g_{20*})((-4 + 9e_{31} - 6e_{41})g_{10*} \\
&+ (-4e_{11} + 3(e_{21} + 3e_{31} - 2e_{41}))g_{20*})) \\
&+ 8\delta((-5 + 10e_{31} + 3e_{41})g_{10*} - (5e_{11} + e_{21} - 10e_{31} - 3e_{41})g_{20*})) \\
&+ 2d(-((g_{10*} + g_{20*})((-1 + 6e_{41})g_{10*} - (e_{11} + 3e_{21} - 6e_{41})g_{20*})) \\
&+ 16\delta((-1 + 3e_{31} - e_{41})g_{10*} - (e_{11} - e_{21} - 3e_{31} + e_{41})g_{20*})) \\
&+ d^2((g_{10*} + g_{20*})((-15 + 34e_{31})g_{10*} + (-15e_{11} + 5e_{21} + 34e_{31})g_{20*}) \\
&+ 16\delta((-4 + 9e_{31} + 2e_{41})g_{10*} \\
&+ (-4e_{11} + 9e_{31} + 2e_{41})g_{20*})) + d^4(8\delta(-g_{10*} + 2e_{31}g_{10*} - e_{11}g_{20*} + 2e_{31}g_{20*}) \\
&- (g_{10*} + g_{20*})((-10 + 15e_{31} + 8e_{41})g_{10*} \\
&+ (-10e_{11} - 3e_{21} + 15e_{31} + 8e_{41})g_{20*}))), \\
g_{11d} &= 2d(4 + d)(-15d^6(g_{10*} + g_{20*})^2 + 6d^7(g_{10*} + g_{20*})^2 \\
&+ 2(g_{10*} + g_{20*})(16\epsilon - 3(g_{10*} + g_{20*})) + 4d^4(\epsilon(g_{10*} - 2g_{20*}) \\
&+ 6(g_{10*} + g_{20*})^2 + 3\delta(2g_{10*} + g_{20*})) + d^5(3(g_{10*} + g_{20*})^2 + 12\delta(2g_{10*} + g_{20*}) \\
&- 4\epsilon(g_{10*} + 4g_{20*})) - 4d^3(6(g_{10*} + g_{20*})^2 - 3\epsilon(g_{10*} + 4g_{20*}) \\
&+ \delta(8\epsilon + 9(2g_{10*} + g_{20*}))) + d(15(g_{10*} + g_{20*})^2 - 8\epsilon(g_{10*} + 4g_{20*}) \\
&+ \delta(-128\epsilon + 24(2g_{10*} + g_{20*}))) - d^2(4\delta(32\epsilon + 6g_{10*} + 3g_{20*}) \\
&+ 3((g_{10*} + g_{20*})^2 + 4\epsilon(3g_{10*} + 2g_{20*})))), \\
g_{21n} &= -((-1 + d^2)(-3(-4 - d + 4d^2 + d^3)g_{10*}(2(-2 + d^2)g_{10*} \\
&- (4 - 3d + d^2)g_{20*})) \times \\
&\times ((6e_{31} + d^2(-1 + 2e_{31}) - 2(1 + e_{41}) + 3d(-1 + 2e_{31} + e_{41}))g_{10*} \\
&- ((2 + 3d + d^2)e_{11} \\
&+ (-2 + d)e_{21} - 6e_{31} - 6de_{31} - 2d^2e_{31} + 2e_{41} - 3de_{41})g_{20*}) \\
&+ d(4 + d)((-4 + 6e_{31} + d^2(-2 + 3e_{31}) + 6e_{41} + d(-8 + 12e_{31} + 3e_{41}))g_{10*}^2 \\
&+ (2 + d^2(1 - 2e_{11}) - 4e_{11} - 4e_{21} - 6e_{31} + d(1 - 8e_{11} - 2e_{21} + 6e_{31} - 3e_{41}) \\
&+ 18e_{41})g_{10*}g_{20*} + ((2 + d + d^2)e_{11} + (-10 + d)e_{21} - 3(4e_{31} + 2de_{31} + d^2e_{31}
\end{aligned}$$

$$\begin{aligned}
& - 4e_{41} + 2de_{41}))g_{20*}^2)(-8(2+d)\epsilon + 3(-1+d)^2(1+d)(2g_{10*} + g_{20*}))))), \\
g_{21d} &= 3(-1+d)^2d(4+5d+d^2)(-4-d+4d^2+d^3)g_{10*}(2(-2+d^2)g_{10*} \\
& - (4-3d+d^2)g_{20*}) + d(4+d)^2(-8(2+d)\epsilon \\
& + 3(-1+d)^2(1+d)(2g_{10*} + g_{20*}))(d^2(8\delta + 3g_{10*}) - 4(g_{10*} + g_{20*}) \\
& - 3d^3(g_{10*} + 2g_{20*}) + d^4(g_{10*} + 4g_{20*}) + d(16\delta + 3g_{10*} + 6g_{20*})), \\
g_{12n} &= 3(-1+d^2)g_{10*}(3d^5(g_{10*} + g_{20*})((-1+3e_{32}+2e_{42})g_{10*} \\
& - (2e_{12}+e_{22}-3e_{32}-2e_{42})g_{20*}) - 8(g_{10*} + g_{20*})((1+3e_{32}-e_{42})g_{10*} \\
& - (e_{12}-e_{22}-3e_{32}+e_{42})g_{20*}) \\
& + d^6(g_{10*} + g_{20*})(-3e_{12}g_{20*} + 5e_{32}(g_{10*} + g_{20*})) \\
& + d^3(-((g_{10*} + g_{20*})(3(1+3e_{32}-2e_{42})g_{10*} \\
& + (-4e_{12}+3(e_{22}+3e_{32}-2e_{42}))g_{20*})) \\
& + 8\delta((-1+10e_{32}+3e_{42})g_{10*} - (5e_{12}+e_{22}-10e_{32}-3e_{42})g_{20*})) \\
& + 2d(-((g_{10*} + g_{20*})((-3+6e_{42})g_{10*} - (e_{12}+3e_{22}-6e_{42})g_{20*})) \\
& + 16\delta((1+3e_{32}-e_{42})g_{10*} - (e_{12}-e_{22}-3e_{32}+e_{42})g_{20*})) \\
& + d^4(-((g_{10*} + g_{20*})((-3+15e_{32}+8e_{42})g_{10*} \\
& + (-10e_{12}-3e_{22}+15e_{32}+8e_{42})g_{20*})) \\
& + 8\delta(-(e_{12}g_{20*}) + 2e_{32}(g_{10*} + g_{20*}))) + d^2((g_{10*} + g_{20*})((5+34e_{32})g_{10*} \\
& + (-15e_{12}+5e_{22}+34e_{32})g_{20*}) + 16\delta(9e_{32}(g_{10*} + g_{20*}) \\
& + 2(-2e_{12}g_{20*} + e_{42}(g_{10*} + g_{20*}))))), \\
g_{12d} &= g_{11d}, \\
g_{22n} &= -((-1+d^2)(-3(-4-d+4d^2+d^3)g_{10*}(2(-2+d^2)g_{10*} \\
& - (4-3d+d^2)g_{20*})((2+6e_{32}+2d^2e_{32}-2e_{42}+d(-1+6e_{32}+3e_{42}))g_{10*} \\
& - ((2+3d+d^2)e_{12} + (-2+d)e_{22} \\
& - 6e_{32}-6de_{32}-2d^2e_{32}+2e_{42}-3de_{42})g_{20*})) \\
& + d(4+d)((-4+6e_{32}+3d^2e_{32}+6e_{42}+d(-2+12e_{32}+3e_{42}))g_{10*}^2 \\
& - (10+4e_{12}+2d^2e_{12}+4e_{22}+6e_{32}-18e_{42} \\
& + d(-1+8e_{12}+2e_{22}-6e_{32}+3e_{42}))g_{10*}g_{20*} \\
& + ((2+d+d^2)e_{12} + (-10+d)e_{22} - 3(4e_{32}+2de_{32}+d^2e_{32} \\
& - 4e_{42}+2de_{42}))g_{20*}^2)(-8(2+d)\epsilon + 3(-1+d)^2(1+d)(2g_{10*} + g_{20*}))))), \\
g_{22d} &= g_{21d}, \\
e_{11n} &= (g_q - g_s p_2)(g_p g_s(m_4 n_2 - m_3 n_3)p_1 \\
& + g_{10*} g_o((m_4 n_1 + m_1 n_3)p_4 - (m_3 n_1 + m_1 n_2)p_5)), \\
e_d &= g_s^3(m_4 n_2 - m_3 n_3)p_1 p_2 + g_{20*} g_o g_q(-(m_4 n_1 + m_1 n_3)p_4 + (m_3 n_1 + m_1 n_2)p_5) \\
& + g_{20*} g_o g_s((m_1 n_3 p_2 + m_4 n_1(-p_1 + p_2))p_4 + m_3 n_1 p_1 p_5 - m_3 n_1 p_2 p_5 \\
& - m_1 n_2 p_2 p_5 + m_2 p_1(n_3 p_4 - n_2 p_5)) + g_k(g_q g_s(-(m_4 n_2) + m_3 n_3) \\
& + g_{20*} g_o(m_4 n_1 p_4 - m_2 n_3 p_4 - m_3 n_1 p_5 + m_2 n_2 p_5)),
\end{aligned}$$

$$\begin{aligned}
e_{12n} &= (g_q - g_s p_2)(g_p g_s(m_4 n_2 - m_3 n_3)p_2 \\
&\quad + g_{10*} g_o(-m_4 n_1 p_4 + m_2 n_3 p_4 + m_3 n_1 p_5 - m_2 n_2 p_5)), \\
e_{21n} &= (g_k - g_s p_1)(g_p g_s(m_4 n_2 - m_3 n_3)p_1 \\
&\quad + g_{10*} g_o(m_4 n_1 p_4 + m_1 n_3 p_4 - m_3 n_1 p_5 - m_1 n_2 p_5)), \\
e_{22n} &= (g_k - g_s p_1)(g_p g_s(m_4 n_2 - m_3 n_3)p_2 \\
&\quad + g_{10*} g_o(-m_4 n_1 p_4 + m_2 n_3 p_4 + m_3 n_1 p_5 - m_2 n_2 p_5)), \\
e_{31n} &= g_{20*} g_p g_s p_1(-g_k m_4 n_1 + g_q m_4 n_1 + g_q m_1 n_3 + g_k m_2 n_3 + g_s m_4 n_1 p_1 \\
&\quad - g_s m_2 n_3 p_1 - g_s m_4 n_1 p_2 - g_s m_1 n_3 p_2) + g_{10*}(g_s p_1(g_s^2(m_4 n_1 + m_1 n_3)p_2 \\
&\quad - g_{20*} g_o(m_1 + m_2)n_1 p_5) + g_k(-(g_q g_s(m_4 n_1 + m_1 n_3)) \\
&\quad + g_{20*} g_o(m_1 + m_2)n_1 p_5)), \\
e_{32n} &= g_{20*} g_p g_s p_2(-g_k m_4 n_1 + g_q m_4 n_1 + g_q m_1 n_3 + g_k m_2 n_3 + g_s m_4 n_1 p_1 \\
&\quad - g_s m_2 n_3 p_1 - g_s m_4 n_1 p_2 - g_s m_1 n_3 p_2) + g_{10*}(g_k g_q g_s(m_4 n_1 - m_2 n_3) \\
&\quad + g_s^3(-m_4 n_1 + m_2 n_3)p_1 p_2 - g_{20*} g_o g_q(m_1 + m_2)n_1 p_5 \\
&\quad + g_{20*} g_o g_s(m_1 + m_2)n_1 p_2 p_5), \\
e_{41n} &= g_{20*} g_p g_s p_1(-g_k m_3 n_1 + g_q m_3 n_1 + g_q m_1 n_2 + g_k m_2 n_2 + g_s m_3 n_1 p_1 \\
&\quad - g_s m_2 n_2 p_1 - g_s m_3 n_1 p_2 - g_s m_1 n_2 p_2) + g_{10*}(g_s p_1(g_s^2(m_3 n_1 + m_1 n_2)p_2 \\
&\quad - g_{20*} g_o(m_1 + m_2)n_1 p_4) + g_k(-(g_q g_s(m_3 n_1 + m_1 n_2)) \\
&\quad + g_{20*} g_o(m_1 + m_2)n_1 p_4)), \\
e_{42n} &= g_{20*} g_p g_s p_2(-g_k m_3 n_1 + g_q m_3 n_1 + g_q m_1 n_2 + g_k m_2 n_2 + g_s m_3 n_1 p_1 \\
&\quad - g_s m_2 n_2 p_1 - g_s m_3 n_1 p_2 - g_s m_1 n_2 p_2) + g_{10*}(g_k g_q g_s(m_3 n_1 - m_2 n_2) \\
&\quad + g_s^3(-m_3 n_1 + m_2 n_2)p_1 p_2 - g_{20*} g_o g_q(m_1 + m_2)n_1 p_4 \\
&\quad + g_{20*} g_o g_s(m_1 + m_2)n_1 p_2 p_4),
\end{aligned}$$

where

$$\begin{aligned}
l_1 &= 24 + 16d - 22d^2 - 16d^3 - 2d^4, \\
m_1 &= 48 - 16d - 52d^2 + 16d^3 + 4d^4, \\
m_2 &= -48 - 80d + 60d^2 + 96d^3 - 10d^4 - 16d^5 - 2d^6, \\
m_3 &= -48 + 112d + 32d^2 - 130d^3 + 14d^4 + 18d^5 + 2d^6, \\
m_4 &= 48 + 104d - 62d^2 - 127d^3 + 11d^4 + 23d^5 + 3d^6, \\
n_1 &= 48 + 56d - 40d^2 - 56d^3 - 8d^4, \\
n_2 &= 48 + 104d - 32d^2 - 104d^3 - 16d^4, \\
n_3 &= -48 + 16d + 10d^2 - 41d^3 + 35d^4 + 25d^5 + 3d^6, \\
o_1 &= 26 - 7d - 27d^2 + 7d^3 + d^4, \\
o_2 &= -12 + 12d^2, \\
p_1 &= -96 - 64d + 88d^2 + 64d^3 + 8d^4, \\
p_2 &= -96 - 64d + 124d^2 + 82d^3 - 26d^4 - 18d^5 - 2d^6,
\end{aligned}$$

$$\begin{aligned}
p_3 &= 96 + 40d - 140d^2 - 60d^3 + 42d^4 + 20d^5 + 2d^6, \\
p_4 &= 144 + 96d - 132d^2 - 96d^3 - 12d^4, \\
p_5 &= 144 + 96d - 186d^2 - 123d^3 + 39d^4 + 27d^5 + 3d^6, \\
p_6 &= -24d - 52d^2 + 4d^3 + 50d^4 + 20d^5 + 2d^6, \\
p_7 &= 96 + 16d - 192d^2 - 56d^3 + 92d^4 + 40d^5 + 4d^6, \\
q_1 &= 96 + 16d - 156d^2 - 38d^3 + 58d^4 + 22d^5 + 2d^6, \\
r_1 &= 24 - 4d - 36d^2 + 2d^3 + 12d^4 + 2d^5, \\
r_2 &= 12 + 2d - 18d^2 - 3d^3 + 6d^4 + d^5, \\
r_3 &= 12 - 6d - 18d^2 + 5d^3 + 6d^4 + d^5, \\
g_s &= g_{10*} + g_{20*}, \\
g_p &= g_{10*} + g_{10*}^2/g_{20*}, \\
g_k &= (g_{10*}^2 p_3)/g_{20*} + g_{20*} p_6 + g_{10*} p_7, \\
g_q &= (-(dg_{20*}^2 l_1) + g_{10*}(g_{10*} p_3 + g_{20*} q_1))/g_{20*}, \\
g_o &= g_s^2/g_{20*}.
\end{aligned}$$

Stability of the fixed point is determined by the  $(2 \times 2)$  block of the stability matrix which corresponds to  $\beta$ -functions of  $\alpha_5$  and  $\chi_3$ . The eigenvalue which responds for instability has the form:

$$\lambda = \lambda_0 + \lambda_1 \alpha_1 + \lambda_2 \alpha_2,$$

where

$$\begin{aligned}
\lambda_0 &= \frac{dg_{20*}(g_{10*} + g_{20*})o_1 - \sqrt{t_1} + g_{10*}g_{20*}r_1 + g_{10*}^2 r_2 + g_{20*}^2 r_3}{8d(12 + 8d^2 + d^2)g_{20*}}, \\
\lambda_1 &= \frac{\lambda_{1n}}{\lambda_d}, \\
\lambda_{1n} &= dg_{20*}^2(g_{11*} + g_{21*})o_1\sqrt{t_1} + g_{20*}(-t_2 + g_{11*}\sqrt{t_1}(g_{20*}r_1 + 2g_{10*}r_2)) \\
&\quad + g_{21*}\sqrt{t_1}(\sqrt{t_1} - g_{10*}^2 r_2 + g_{20*}^2 r_3), \\
\lambda_d &= 8d(12 + 8d + d^2)g_{20*}^2\sqrt{t_1}, \\
\lambda_2 &= \frac{\lambda_{2n}}{\lambda_d}, \\
\lambda_{2n} &= dg_{20*}^2(g_{12*} + g_{22*})o_1\sqrt{t_1} + g_{20*}(-t_3 + g_{12*}\sqrt{t_1}(g_{20*}r_1 + 2g_{10*}r_2)) \\
&\quad + g_{22*}\sqrt{t_1}(\sqrt{t_1} - g_{10*}^2 r_2 + g_{20*}^2 r_3),
\end{aligned}$$

with

$$\begin{aligned}
t_1 &= d^2 g_{20*}^2 (g_{10*} + g_{20*})^2 (o_1^2 - 4o_2^2) - 2dg_{20*}o_1(g_{10*} + g_{20*}) \times \\
&\quad \times (g_{10*}g_{20*}r_1 + g_{10*}^2 r_2 + g_{20*}^2 r_3) + (g_{10*}g_{20*}r_1 + g_{10*}^2 r_2 + g_{20*}^2 r_3)^2,
\end{aligned}$$

$$\begin{aligned}
t_2 &= 2(d^2 g_{20*}(g_{10*} + g_{20*})(g_{11*}g_{20*} + (g_{10*} + 2g_{20*})g_{21*})(o_1^2 - 4o_2^2) \\
&+ (g_{10*}g_{20*}r_1 + g_{10*}^2r_2 + g_{20*}^2r_3)(g_{11*}g_{20*}r_1 + g_{10*}g_{21*}r_1 + 2g_{10*}g_{11*}r_2 \\
&+ 2g_{20*}g_{21*}r_3) - do_1(g_{10*}^3g_{21*}r_2 + g_{10*}^2g_{20*}(3g_{11*}r_2 + 2g_{21*}(r_1 + r_2)) \\
&+ g_{20*}^3(4g_{21*}r_3 + g_{11*}(r_1 + r_3)) + g_{10*}g_{20*}^2(2g_{11*}(r_1 + r_2) + 3g_{21*}(r_1 + r_3))))), \\
t_3 &= 2(d^2 g_{20*}(g_{10*} + g_{20*})(g_{12*}g_{20*} + (g_{10*} + 2g_{20*})g_{22*})(o_1^2 - 4o_2^2) \\
&+ (g_{10*}g_{20*}r_1 + g_{10*}^2r_2 + g_{20*}^2r_3)(g_{12*}g_{20*}r_1 + g_{10*}g_{22*}r_1 + 2g_{10*}g_{12*}r_2 \\
&+ 2g_{20*}g_{22*}r_3) - do_1(g_{10*}^3g_{22*}r_2 + g_{10*}^2g_{20*}(3g_{12*}r_2 + 2g_{22*}(r_1 + r_2)) \\
&+ g_{20*}^3(4g_{22*}r_3 + g_{12*}(r_1 + r_3)) + g_{10*}g_{20*}^2(2g_{12*}(r_1 + r_2) + 3g_{22*}(r_1 + r_3))))).
\end{aligned}$$

borderline dimension  $d_c$  is defined as a solution of the equation

$$\lambda(d_c, \epsilon, \alpha_1, \alpha_2) = 0.$$

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